# LINEAR METHODS FOR TENSORS
# LECTURE #1 (CENTROIDS)

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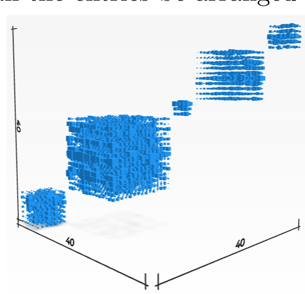
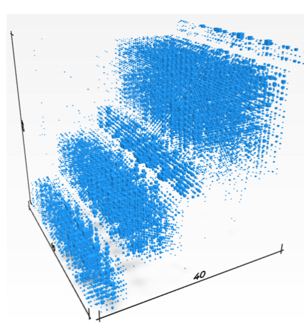
## 1. Overview & Objectives

We learned in previous lectures what it is that makes a tensor a tensor. We have seen that, roughly speaking, the study of tensors is concerned with properties of "$n$-way arrays" of numbers up to basis changes on the "axes" of the arrays. This series of lectures is motivated, in part, by the desire to answer certain computational questions about tensors:

(1) Is it possible to reveal, by a basis change, structure that exists in a given tensor? For instance, can the entries be arranged into diagonal blocks




or perhaps into "steps" relative to some face of the tensor:

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(2) Is it possible to decide, by a basis change, whether some tensor can be transformed into another? (The so-called *Tensor Isomorphism Problem.*)

These lectures introduce linear invariants of tensors, namely invariants that can be computed as solutions to systems of linear equations. We will focus on invariants that are naturally equipped with some algebraic structure. We will show that problems of the first type can be solved efficiently using algebraic properties of the linear invariants. We will also lay some groundwork for the tensor isomorphism problem and discuss some special cases, but this is a topic that will be discussed in much more detail in the series of lectures by Joshua Maglione.

There are computer implementations of algorithms based on ideas and techniques outlined in these notes. Throughout these notes I will provide illustrations of computations that can be carried out on the Magma system [1] using supplementary packages that are freely available as GitHub repositories at [2].

## 2. Notation

Following the notation established in the lectures by James Wilson and Elina Rebeva, our main object of study will be a bimap (3-tensor)

$$* : U \times V \rightarrowtail W, \qquad \text{where } U, V, W \text{ are abelian groups.}$$

Recall that the symbol $\rightarrowtail$ communicates that $*$ distributes over the addition in $U$ and $V$. The techniques developed in this series of lectures apply more generally to $n$-tensors for any $n \geqslant 3$, but bimaps are general enough to adequately illustrate their power. Confining our attention to bimaps will make the notation more transparent and the applications easier to visualize.

2.1. **Systems of forms.** It is often convenient to appeal to matrices when computing with, or reasoning about linear transformations (2-tensors). Let's recall how this works. Let $T : U \to V$ be a linear transformation of $\mathbb{F}$-vector spaces, where $\dim U = c$ and $\dim V = d$. Fix ordered bases $\{u_1, \ldots, u_c\}$ and $\{v_1, \ldots, v_d\}$ for $U$ and $V$, respectively. Now associate to $T$ the $c \times d$ matrix $M$ whose $i$th row consists of the coordinates of the vector $T(u_i)$ written relative to $\{v_1, \ldots, v_d\}$.

We saw in earlier lectures that higher-valence tensors may be similarly represented as grids (multiway arrays) of scalars. In the case of bimaps, though, we often prefer the visual convenience of 'flattening' to a *system of forms*. Given $* : U \times V \rightarrowtail W$, fix ordered bases $\{u_1, \ldots, u_c\}$, $\{v_1, \ldots, v_d\}$ and $\{w_1, \ldots, w_e\}$ for $U, V$ and $W$, respectively. Now form the list $[M_1, \ldots, M_e]$ of $c \times d$ matrices, where the $(i, j)$th entry of $M_k$ is $\alpha_{ijk}$, where $u_i * v_j = \alpha_{ij1} w_1 + \ldots + \alpha_{ije} w_e$.

If $[M_1, \ldots, M_e]$ is the system of forms representing a bimap $* : U \times V \rightarrowtail W$ relative to some fixed bases, then we have

$$\forall u \in U, \forall v \in V, \qquad u * v = (\, uM_1 v^\top \,, \ldots \,, \, uM_e v^\top \,) \tag{1}$$

as a row vector relative to the fixed basis of $W$, where the $u$'s and $v$'s on the right of the equality are row vectors relative to the fixed bases of $U$ and $V$, respectively.

*Whenever we represent bimaps as systems of forms,*
*we assume we are working with $\mathbb{F}$-bimaps for some field $\mathbb{F}$.*

2.2. **Degeneracy.** Recall from earlier lectures that, if

$$V^{\top} = \{u \in U \mid u * V = 0\} \qquad U^{\perp} = \{v \in V \mid U * v = 0\}, \tag{2}$$

then $* : U \times V \rightarrowtail W$ induces a bimap

$$\begin{tikzcd} U \times V \arrow[r, tail, "*"] \arrow[d, two heads] & W \\ U/V^{\top} \times V/U^{\perp} \arrow[r, tail, "\bullet"] & U * V \arrow[u, hook] \end{tikzcd}$$

where no information is lost. We therefore assume throughout these lectures that

$\boxed{* : U \times V \rightarrowtail W \text{ is } \textbf{fully nondegenerate}\text{: } V^{\top} = 0;\ U^{\perp} = 0;\ U * V = W}$

**Example 2.1.** *Here is an illustration of how one can pass to a nondegenerate tensor using Magma. First, we build a bimap (as a 3-tensor) with degeneracies:*

```
> S := [ Matrix (Rationals(), 2, 3, [1,0,0,0,0,0]),
>        Matrix (Rationals(), 2, 3, [0,0,0,0,1,0]) ];
> t := Tensor (S, 2, 1);
> SystemOfForms (t);
[
    [1 0 0]
    [0 0 0],

    [0 0 0]
    [0 1 0]
]
```

*We can see from the system of forms that the tensor is degenerate: picturing it as a cube of data, the entire right side of the cube—which corresponds to basis vector $\langle 0, 0, 1\rangle$ of $V$—consists of 0s. Let's confirm our observation:*

```
> Radical (t);
<
    Vector space of degree 2, dimension 0 over Rational Field,

    Vector space of degree 3, dimension 1 over Rational Field
    Echelonized basis:
    (0 0 1)
>
```

*Now, let's ask Magma to compute the induced nondegenerate tensor $s$:*

```
> s := NondegenerateTensor (t);
> SystemOfForms (s);
[
    [1 0]
    [0 0],

    [0 0]
    [0 1]
]
```

*Using our notation, we can see that, for the original tensor, $V^{\top}$ is 0-dimensional, $U^{\perp}$ is 1-dimensional, and $U*V = W$. Inspection of the system of forms for $s$ reveals that Magma has, as we would expect, simply lopped off the right face of 0s.*

## 3. The centroid of a bimap

We have only assumed so far that $U, V, W$ are abelian groups, and that $* : U \times V \rightarrowtail W$ distributes over the addition in $U$ and $V$. However, familiar instances of bimaps also play nicely with scalars. One such is the dot product on $\mathbb{R}^n$, where

$$\forall u \in \mathbb{R}^n,\ \forall v \in \mathbb{R}^n,\ \forall \alpha \in \mathbb{R}, \qquad (\alpha u) * v = u * (\alpha v) = \alpha(u * v).$$

By thinking of $\alpha$ as an endomorphism of the abelian groups $\mathbb{R}^n$ and $\mathbb{R}$, we can generalize the notion of scalars playing nicely with bimaps.

Let us view an abelian group $U$ as a right $\mathrm{End}(U)$-module, writing $ux$ for the image of $u \in U$ under $x \in \mathrm{End}(U)$. For $\Omega \subseteq \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W)$, say that $* : U \times V \rightarrowtail W$ is an $\Omega$-*bimap* if

$$\forall u \in U,\ \forall v \in V,\ \forall (x, y, z) \in \Omega, \qquad (ux) * v = u * (vy) = (u * v)z. \tag{3}$$

Without any further qualifications, any bimap $* : U \times V \rightarrowtail W$ is a $\mathbb{Z}$-bimap, but it makes sense to work with the *largest* set $\Omega$ of scalars for which $*$ is an $\Omega$-bimap:

**Definition.** The *centroid* of a bimap $* : U \times V \rightarrowtail W$ is the set

$$\mathrm{Cen}(*) = \{\ (x, y, z) \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) \mid \forall u \in U, \forall v \in V,\ ux * v = u * vy = (u * v)z\ \}.$$

3.1. **Actions on matrices.** Let's reformulate the centroid condition in terms of matrices. Fixing bases of $U, V, W$, we can represent the linear transformations $x, y, z$ by matrices $X, Y, Z$. Referring to (1), we have

$$\forall u \in U, \forall v \in V \qquad \begin{aligned} ux * v &= (\ uXM_1v^\top\ , \dots ,\ uXM_ev^\top\ ) \\ u * vy &= (\ uM_1Y^\top v^\top\ , \dots ,\ uM_eY^\top v^\top\ ) \\ (u * v)z &= (\ uM_1v^\top\ , \dots ,\ uM_ev^\top\ )Z \end{aligned}$$

For obvious reasons the first two actions are referred to as 'inner' and the third as 'outer'. The condition $ux * v = u * vy$ translates directly to the matrix equations

$$\forall i \in \{1, \dots, e\} \qquad XM_i = M_iY^\top \tag{4}$$

The condition $ux * v = (u * v)z$ also translates into a linear system:

$$\forall i \in \{1, \dots, e\} \qquad XM_i = \sum_{j=1}^{e} M_jZ_{ji} \tag{5}$$

The point, in case it was not already obvious from the definition of $\mathrm{Cen}(*)$, is that the centroid can be computed as the solution of a system of linear equations. To be more precise, it is the solution of a system of $cde$ equations in the $c^2 + d^2 + e^2$ unknown entries of the matrices $X, Y, Z$. *The centroid can be computed efficiently!*

3.2. **Algebra of centroids.** On the face of it, $\mathrm{Cen}(*)$ is just a *set* of operators. Well OK, since $(0, 0, 0) \in \mathrm{Cen}(*)$ and $\mathrm{Cen}(*)$ is the solution of a linear system, it is a *linear subspace* of $\mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W)$. However, the universe decrees that $\mathrm{Cen}(*)$ possesses a richer algebraic structure:

**Theorem 3.1.** *The centroid of a bimap is a commutative ring.*

*Proof.* Both commutativity and closure of the product on $\mathrm{Cen}(*)$ follow from the '3-pile shuffle' argument made in previous lectures. Let us review the argument.

Let $(x, y, z), (\acute{x}, \acute{y}, \acute{z}) \in \mathrm{Cen}(*)$. For $u \in U$, $v \in V$, we have

$$(ux\acute{x}) * v = ((ux) * v)\acute{z} = (u * (vy))\acute{z} = \begin{cases} u * (vy\acute{y}) \\ (u * v)z\acute{z} \end{cases}, \tag{6}$$

so $(x\acute{x}, y\acute{y}, z\acute{z}) \in \mathrm{Cen}(*)$. Also, for $u \in U$, $v \in V$,

$$(ux\acute{x}) * v = ((ux) * v)\acute{z} = (u * (vy))\acute{z} = (u\acute{x}) * (vy) = (u\acute{x}x) * v, \tag{7}$$

so $u(x\acute{x} - \acute{x}x) * v = 0$. As $*$ is nondegenerate, it follows that $x\acute{x} = \acute{x}x$. Similarly for the actions on the other coordinates.

Closure of $\mathrm{Cen}(*)$ under addition is left as an exercise. $\square$

**Example 3.1.** *Computing the centroid is really the first step when studying a given bimap $* : U \times V \rightarrowtail W$ because it helps us position $*$ in its natural environment. Here is an illustatrion of how to use Magma to compute the correct bimap:*

```
> S := [ Matrix(GF(8191), 4, 4,
>        [ 2759, 2012, 3809, 4443, 5905, 5577, 6247, 7780,
>          4853, 7986, 7596, 3247, 2328, 243, 352, 6905 ]),
>        Matrix(GF(8191), 4, 4,
>        [ 7025, 2820, 3922, 7714, 5317, 4257, 3882, 2430,
>          1804, 6985, 3119, 4676, 4033, 4053, 2850, 5866 ]) ];
> t := Tensor (S, 2, 1);
> C := Centroid (t);
> Dimension (C);
2
> s := TensorOverCentroid (t);
> s;
Tensor of valence 3, U2 x U1 >-> U0
U2 : Full Vector space of degree 2 over GF(8191^2)
U1 : Full Vector space of degree 2 over GF(8191^2)
U0 : Full Vector space of degree 1 over GF(8191^2)
```

*The tensor $t : \mathbb{F}_{8191}^4 \times \mathbb{F}_{8191}^4 \rightarrowtail \mathbb{F}_{8191}^2$ is built using a random-looking[1] pair of $4 \times 4$ matrices. We see that the centroid $C = \mathrm{Cen}(t)$ is a 2-dimensional $\mathbb{F}_{8191}$-algebra, so it could be $\mathbb{F}_{8191} \times \mathbb{F}_{8191}$, or it could be the field extension $\mathbb{F}_{8191^2}$. Magma can help us determine which one it is, but it turns out to be the field extension. Now, we use the function* `TensorOverCentroid` *to write $t$ as a $\mathbb{F}_{8191}$-bimap. We see, in fact, that $t$ is a bilinear $\mathbb{F}_{8191^2}$-form in disguise!*

3.3. **Diagonal blocks.** Is it possible, in an example such as the last one, to encounter a 2-dimensional centroid whose structure is $\mathbb{F}_{8191} \times \mathbb{F}_{8191}$ rather than $\mathbb{F}_{8191^2}$? And how could we use the centroid to reveal the structure of the bimap? We saw such an example in an earlier lecture. Let's revisit that example now, and analyze it using the centroid.

**Example 3.2.** *I am referring to the 'multiplication table' example. Define a bimap $* : \mathbb{Q}^2 \times \mathbb{Q}^2 \rightarrowtail \mathbb{Q}^2$ using the following pair of matrices:*

$$\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$$

[1] Spoiler alert—it is not actually random! It was, of course, intentionally built to have a nontrivial centroid. The code used to build it can be found in the accompanying files.

*Earlier, we saw that we can interpret this bimap as the multiplication in the quotient ring* $\mathbb{Q}[x]/(x+1)(x+2)$*. The following Magma session shows how this structure is revealed using the centroid.*

```
> Q := Rationals();
> S := [ Matrix (Q, 2, 2, [1,0,0,-2]),
>        Matrix (Q, 2, 2, [0,1,1,-3]) ];
> t := Tensor (S, 2, 1);
> C := Centroid (t);
> Dimension(C);
2
> C.2;
[ 0  1  0  0  0  0]
[-2 -3  0  0  0  0]
[ 0  0  0  1  0  0]
[ 0  0 -2 -3  0  0]
[ 0  0  0  0  0  1]
[ 0  0  0  0 -2 -3]
> Factorization (MinimalPolynomial (C.2));
[
    <$.1 + 1, 1>,
    <$.1 + 2, 1>
]
```

*We see that* $\mathrm{Cen}(*)$ *is again a 2-dimensional algebra. The first generator is the identity. The second generator,* `C.2`*, is displayed as the diagonal join of its actions on* $U$*,* $V$ *and* $W$*. Its minimal polynomial is* $(x+1)(x+2)$*, as anticipated. We can continue using Magma to diagonalize the individual blocks, and then use that basis to rewrite bimap:*

```
> X := MatrixAlgebra (Q, 2) ! Matrix(Q, 2, 2, [2,1,1,1]);
> Y := Transpose (X);
> Z := Transpose (X^-1);
> outer_action ([ X * s * Y : s in S ], Z);
[
    [1 0]
    [0 0],

    [ 0  0]
    [ 0 -1]
]
```

*In the interest of space, the construction of the change-of-basis matrices* $X, Y, Z$ *in Magma has been omitted, but rest assured it was basic linear algebra. The outcome is the same as in the original multiplication table when we used '*$x+2$*' and '*$x+1$*' to label the rows and columns, namely we have discovered a diagonal block decomposition of the bimap.*

The example shows that block structures within tensor data correspond to algebraic properties of linear invariants of the tensor. We will return to this idea repeatedly, and in more detail, as we continue along our path.

3.4. **The centroid of a general tensor.** As noted earlier, we are focusing in these lectures on bimaps (3-tensors) largely for reasons of notational and visual convenience, but when it is appropriate we will explain how to define our algebraic invariants for tensors of arbitrary valence. For this, we will again follow the notation set up in the lectures by Elina Rebeva and James Wilson.

If $T$ is a tensor space with interpretation map

$$\langle\ |\ \rangle : T \times \left(\prod_{a\in A} V_a\right) \to \Delta,$$

the centroid of $t \in T$ is the set

$$\mathrm{Cen}(t) = \left\{ \omega \in \prod_{a\in A} \mathrm{End}(V_a) \;\middle|\; \forall a,b \in A,\ \langle t|v_a\omega_a, v_{\bar{a}}\rangle = \langle t|v_b\omega_b, v_{\bar{b}}\rangle \right\} \tag{8}$$

If $t$ is nondegenerate, then $\mathrm{Cen}(t)$ is a commutative ring.

## References

[1] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **3-4** (1997), 235-265.

[2] Peter Brooksbank, Joshua Maglione, and James Wilson, *The Tensor Space*, `https://github.com/thetensor-space/TensorSpace`.