Spaces of matrices of bounded rank

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A problem in classical linear algebra

$\mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$: space of $\mathbf{b} \times \mathbf{c}$ matrices.

Let $E \subset \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$ be a linear subspace such that

$\forall e \in E$, $\text{rank}(e) < \min\{\mathbf{b}, \mathbf{c}\}$, say E has *bounded rank*. If $\forall e \in E$, $\text{rank}(e) \leq r$, say *bounded rank* r.

Example: $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & 0 & 0 & 0 \\ x_{10} & 0 & 0 & 0 \end{pmatrix}$, bounded rank 3.

Example: $\begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix}$, bounded rank 2.

Classical Q: What are the spaces of bounded rank?

Relation with tensors

Let $A = \mathbb{C}^{\mathbf{a}}, B = \mathbb{C}^{\mathbf{b}}, C = \mathbb{C}^{\mathbf{c}}$, and let $T \in A \otimes B \otimes C$.

$\exists$ 1-1 correspondence

$T \in A \otimes B \otimes C,\ T \sim gT\ \forall g \in GL(A) \times GL(B) \times GL(C)$
$\leftrightarrow$
$E \subset B \otimes C$ of dimension $\mathbf{a}$, $E \sim hE\ \forall h \in GL(B) \times GL(C)$

Send $T \mapsto E := T(A^*)$.

$T(A^*)$ bounded rank $=:$ T is 1_A-*degenerate*

T is 1-*degenerate* if $1_A, 1_B, 1_C$ degenerate

Least understood tensors: quantum information theory, Strassen's laser method for upper bounding exponent of matrix multiplication.

Classical examples

1. Compression spaces: $\exists B' \subset B^*$, $C' \subset C$ with $E(B') \subseteq C'$.
2. skew symmetric matrices of odd size: $\Lambda^2 V \subset V \otimes V$.
3. $V \to \mathrm{Hom}(V, \Lambda^2 V)$, $w \mapsto (v \mapsto w \wedge v)$.

Exercise dim V odd: 2,3 same tensor.

1983: Thm. Atkinson/Llyod, Atkinson: classified bounded rank $r \leq 3$. No non-classical examples.

no progress on classification for 40 years.

1996: Westwick, first non-classical example $r = 8$. Since then, many with large r. In particular, have interesting moduli.

Theorem (Huang-L) 2023

Up to isomorphism, there exist 4 basic spaces (non-compression etc.) of bounded rank 4:

1. $\Lambda^2\mathbb{C}^5 \subset \mathbb{C}^5 \otimes \mathbb{C}^5$
2. $\mathbb{C}^5 \to \mathrm{Hom}(\mathbb{C}^5, \Lambda^2\mathbb{C}^5)$
3. $$\begin{pmatrix} a_1 & & & & -a_3 & -a_5 \\ & a_1 & & & -a_4 & -a_6 \\ & & a_1 & & a_2 & 0 \\ & & & a_1 & 0 & a_2 \\ a_2 & 0 & a_3 & a_5 & 0 & 0 \\ 0 & a_2 & a_4 & a_6 & 0 & 0 \end{pmatrix} \subset \mathbb{C}^6 \otimes \mathbb{C}^6,$$
4. $$\begin{pmatrix} a_1 & a_2 & 0 & 0 & -a_5 & -a_6 \\ 0 & a_1 & 0 & 0 & 0 & -a_5 \\ 0 & 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & a_1 & 0 & a_3 \\ a_3 & a_4 & a_5 & a_6 & 0 & 0 \\ 0 & a_3 & 0 & a_5 & 0 & 0 \end{pmatrix} \subset \mathbb{C}^6 \otimes \mathbb{C}^6.$$

Geometry of case III

In general, $\mathcal{A}$: algebra, get structure tensor $T_{\mathcal{A}} \in \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}$, i.e., $T_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \mapsto ab$.

Four composition algebras $\rightsquigarrow$ four exceptional complex simple Lie algebras.

e.g. quaternions $\rightsquigarrow$ E_7, octonions $\rightsquigarrow$ E_8

L-Manivel 2006: $E_{7\frac{1}{2}}$ $\rightsquigarrow$ sextonions!

Example 3 is structure tensor of the sextonions.

Detour: Strassen's laser method

Exponent ω of matrix multiplication: 1968 $\omega \leq 2.81$ much work, many people 1988
$\omega \leq 2.38$.

since 1988 essentially no improvement.

Strassen: instead of direct upper bounds on $M_{\langle n \rangle}$, use "easy" auxiliary tensors.
Essentially only 2 known such that could potentially prove $\omega = 2$:

Small Coppersmith-Winograd tensor: $T_{cw,2} \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ monomial $a_1 a_2 a_3 \in S^3\mathbb{C}^3$ considered as a tensor, i.e.,
$T_{cw,2} = \sum_{\sigma \in \mathfrak{S}_3} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes a_{\sigma(3)}$

"Skew small CW": $T_{skewcw,2} \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ element of $\Lambda^3\mathbb{C}^3$ considered as a tensor, i.e.,
$T_{skewcw,2} = \sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma) a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes a_{\sigma(3)}$

Would be very happy with something that could prove ≤ 2.37.

Strassen's laser method cont'd

Rank $\mathbf{R}(T)$: smallest r such that T sum of r rank one tensors.
Border rank $\underline{\mathbf{R}}(T)$: smallest r such that T limit of rank r tensors.
Example $W = a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1$, $\mathbf{R}(W) = 3$, $\underline{\mathbf{R}}(W) = 2$, laser method gives good ($\omega \leq 2.4...$) with $T = W$.

Strassen: take high Kronecker power $\boxtimes$ of auxiliary tensor, where $T_1 \boxtimes T_2 = T_1 \otimes T_2 \in (A_1 \otimes A_2) \otimes (B_1 \otimes B_2) \otimes (C_1 \otimes C_2)$.

For example (Coppersmith-Winograd): $\omega \leq \log_2(\frac{4}{27}(\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes k}))^{\frac{3}{k}})$.
$\underline{\mathbf{R}}(T_{cw,2}) = 4 > 3$, if had 3 or $(const.)3^k$ large k would get $\omega = 2$.

Thm (Conner-Gesmundo-L-Ventura): same holds for $T_{skewcw,2}$
Sad news: $\underline{\mathbf{R}}(T_{skewcw,2}) = 5$
More sad news: Thm (Conner-Huang-L) $\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes 2}) = 16 = 4^2$

Hopeful news and Example IV

Thm (Conner-Gesmundo-L-Ventura): $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes 2}) \leq 17 < 5^2$
(in fact = (Conner-Harper-L))

Thm (Huang-L) Example IV is $T_{skewcw,2} \boxtimes W$

and $\underline{\mathbf{R}}(T_{skewcw,2} \boxtimes W) = 9 < 10$.

Opens new path to proving upper bounds on ω.

Ideas towards proof of main theorem: only four basic cases $r = 4$

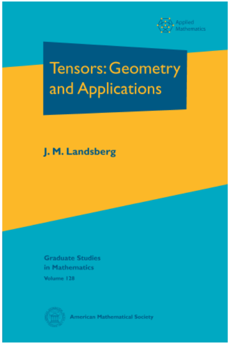
Classical methods: $\dim B, C \leq 7$, many reductions.
Algebraic geometry (Sylvester, Eisenbud-Harris): $T \in A \otimes B \otimes C \rightsquigarrow$

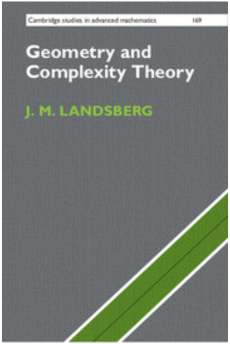
$$\begin{array}{ccc} B^* \otimes \mathcal{O}_{\mathbb{P}A^*} & \longrightarrow & C \otimes \mathcal{O}_{\mathbb{P}A^*}(1) \\ & \searrow \quad \swarrow & \\ & \mathbb{P}A^* & \end{array}$$

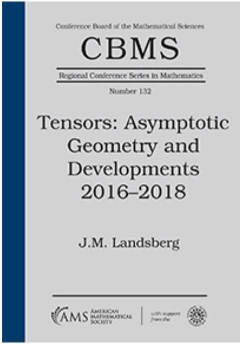
invariants image sheaf.
Commutative algebra:
Buchsbaum-Eisenbud characterization of exact complexes
Hilbert-Burch
Buchsbaum-Eisenbud generalization of HIlbert-Burch to codimension three Gorenstein.
$\rightsquigarrow$ any other potential basic space is skew-symmetrizable.

Thank you for your attention

For more on **tensors**, their geometry and applications, resp. **geometry and complexity**, resp. **asymptotic geometry**:







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